PH 5840 Lecture -10

Note Title 8/31/2015

SVD, Schmidt decomposition and purification

Which state is more entangled?

$$\begin{cases} |\Psi\rangle = \dfrac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} \\ |\Phi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}} \end{cases}$$

Theorem:- Given a pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\exists$ ON states $\{|e_i\rangle_A\} \in \mathcal{H}_A$ and $\{|f_j\rangle_B\} \in \mathcal{H}_B$ such that,

$$|\psi\rangle_{AB} = \sum_{i=1}^{k} \lambda_i |e_i\rangle_A |f_i\rangle_B ,$$

where $\{\lambda_i\}$ are non-negative real nos. Satisfying $\sum_i \lambda_i^2 = 1$

Proof:- Let $\{|j\rangle_A\} \in \mathcal{H}_A$, $\{|k\rangle_B\} \in \mathcal{H}_B$ be ON bases for $\mathcal{H}_A$ and $\mathcal{H}_B$.

$\dim(\mathcal{H}_A) = d_A$
$\dim(\mathcal{H}_B) = d_B$
$d_A \neq d_B$

Then, $$|\psi\rangle_{AB} = \sum_{j=1}^{d_A} \sum_{k=1}^{d_B} c_{jk} |j\rangle_A |k\rangle_B .$$

Use: Singular value decomposition :-

SVD: Any $m\times n$ complex-valued matrix can be decomposed as

$$C = U D V,$$

where $U \equiv m\times m$ unitary matrix

$V \equiv n\times n$ unitary matrix

$D \equiv m \times n$ rectangular diagonal matrix with $r = \min(m,n)$ non-zero entries on the diagonal.

Simple "proof":-

Let $M = \sqrt{C^{+}C}$ ($n \times n$ positive matrix)

$\exists$ unitary V : $V^{+}\sqrt{C^{+}C}\,V = D$ (diagonal)

Lets say D has 'r' nonzero entries.

$$V = \Big(\underbrace{|\ |\ |\ |}_{r}\ \ \Big)_{n \times n}$$

Pick the first 'r' columns of V.

$V = (V_1\ V_2) \Leftarrow$ $V_1 = \Big(\underbrace{|\ |\ |\ |}_{r}\Big)_{n \times r}$ $V_2 = \Big(\underbrace{|\ |\ |\ \ldots}_{r+1 \ldots n}\Big)_{n \times (n-r)}$

$$V_1^{+}\sqrt{C^{+}C}\,V_1 = D_1 \quad , \quad V_2^{+}\sqrt{C^{+}C}\,V_2 = 0$$

Define: $U_1 = C V_1 D_1^{-1}$ ($m \times r$ matrix)

Isometry! $U_1^{+}U_1 = D_1^{-1} \underbrace{V_1^{+} C^{+} C V_1}\, D_1^{-1} = I_{r \times r}$

Add rows to U_1 so that $U = (U_1\ U_2)$ is a $m \times m$ unitary.

$$U D V^{+} = (U_1\ U_2)\begin{pmatrix} D_1 & \\ & 0 \end{pmatrix}\begin{pmatrix} V_1^{+} \\ V_2^{+} \end{pmatrix}$$

$$= U_1 D_1 V_1^{+} = \underline{\underline{C}}$$

* The eigenvalues of $\sqrt{C^{+}C}$ are called the singular values of C.

* Given SVD, $C = U D V^{+}$

$$C^\dagger C = V D U^\dagger U D V^\dagger = V D^2 V^\dagger$$

$$C C^\dagger = U D V^\dagger V D U^\dagger = U D^2 U^\dagger$$

$\therefore$ Eigenvalues of $C^\dagger C$ = evalues of $CC^\dagger$

= sq. of singular values of C.

* columns of V: right singular vectors
  eigenvectors of $C^\dagger C$
* columns of U: left singular vectors
  eigenvectors of $CC^\dagger$

(Simpler proof of SVD:- Section 2.1.10 of Nielsen & Chuang)

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* Back to Schmidt decomposition!

$$|\psi\rangle_{AB} = \sum_{jk} C_{jk} |j\rangle_A |k\rangle_B$$

SVD: $$C_{jk} = \sum_{p=1}^{d_A} \sum_{q=1}^{d_B} u_{jp} d_{pq} v_{qk} \qquad \left(\begin{array}{l} d_{pq} = \delta_{pq} d_{pp} \\ \text{for } 1 \le p, q \le r \\ 0 \text{ otherwise} \end{array} \right)$$

$$\therefore |\psi\rangle_{AB} = \sum_{pq} \sum_{j=1}^{d_A} \sum_{k=1}^{d_B} u_{jp} d_{pq} v_{qk} |j\rangle_A |k\rangle_B$$

$$= \sum_{p,q}^{d_A, d_B} d_{pq} \left[\left(\sum_{j=1}^{d_A} u_{jp} |j\rangle_A \right) \left(\sum_{k=1}^{d_B} v_{qk} |k\rangle_B \right) \right]$$

Define $|e_p\rangle_A \equiv \sum_j u_{jp} |j\rangle_A$, $|f_q\rangle_B \equiv \sum_k v_{qk} |k\rangle_B$

$\langle e_\ell | e_i \rangle = \delta_{i\ell} = \langle f_\ell | f_i \rangle$

by unitarity of U and V.

$$\therefore |\psi\rangle_{AB} = \sum_{p,q} d_{pq} |e_p\rangle_A |f_q\rangle_B = \sum_{p=1}^{r} d_{pp} |e_p\rangle_A |f_p\rangle_B$$

$\sum_i d_{ii} = 1$ follows from $\langle \psi_{AB} | \psi_{AB} \rangle = 1$.

(1) Usefulness of Schmidt decomposition:-

(i) Reduced density operators:-

$$\rho_A = \sum_i \lambda_i^2 |i\rangle\langle i| \qquad \rho_B = \sum_j \lambda_j^2 |j\rangle\langle j|$$

Have identical eigenvalues!

(ii) # of non-zero Schmidt coefficients

$r \equiv$ Schmidt number / Schmidt rank

A measure of how entangled the state is.

$$1 \leq r \leq \min(d_A, d_B)$$

- If Schmidt rank $(|\psi\rangle_{AB}) = 1$, $|\psi\rangle_{AB} = |e_1\rangle_A \otimes |f_1\rangle_B$ product state!
- Any state $|\psi\rangle_{AB}$ with $r > 1$ is entangled!

(iii) Entanglement entropy:-

Given, Schmidt form: $|\psi\rangle_{AB} = \sum_i \lambda_i |i\rangle_A |i\rangle_B$, $\sum_i \lambda_i^2 = 1$

$$\rho_A = \sum_i \lambda_i^2 |i\rangle_A\langle i| \quad \text{and} \quad \rho_B = \sum_i \lambda_i^2 |i\rangle_B\langle i|$$

- $\{\lambda_i^2\}$ is a probability distribution $(0 \leq \lambda_i^2 \leq 1)$

* Shannon entropy: $H(\{\lambda_i^2\}) = -\sum_i \lambda_i^2 \log_2 \lambda_i^2$

$$\equiv H(\rho_A) = H(\rho_B)$$

$$0 \leq H(\rho_A) \leq \log_2 r$$

- Entanglement entropy $\equiv H(\rho_A) = H(\rho_B)$ is a of how entangled $|\psi\rangle_{AB}$ is.

* If $|\psi\rangle_{AB}$ is a product state, $H(\rho_A) = 0 = H(\rho_B)$
$|\psi\rangle_{AB}$ is maximally entangled, $H(\rho_A) = \log_2 d$
$= H(\rho_B)$.

(3) Transforming to the Schmidt basis :-

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A |j\rangle_B$$

$$\rho_A = \sum_{i,j,l} c_{ij}^{*} c_{lj} |l\rangle\langle i|_A = \sum_{i,l} a_{il} |l\rangle\langle i|$$

Diagonalize ρ_A; let $\{|e_i\rangle_A\}$ be the basis in which ρ_A is diagonal.

$$\therefore \rho_A = \sum_i \lambda_i^2 |e_i\rangle\langle e_i|_A$$

Now, $|\psi_{AB}\rangle = \sum_{i,j} \alpha_{ij} |e_i\rangle_A |j\rangle_B$

$= \sum_i |e_i\rangle_A |\tilde{i}\rangle_B$, where,

$$|\tilde{i}\rangle_B = \sum_j \alpha_{ij} |j\rangle_B$$

Partial trace over B

$$\Rightarrow \rho_A = \sum_{i,l} \langle \tilde{l}|\tilde{i}\rangle \, |e_i\rangle\langle e_l|_A$$

$$\Rightarrow \langle \tilde{l}|\tilde{i}\rangle = \lambda_i \delta_{il}$$

$\therefore$ The $\{|\tilde{i}\rangle_B\}$ are orthogonal indeed!

Normalize them: $|f_i\rangle_B = \lambda_i^{-1} |\tilde{i}\rangle_B$

Then, $|\psi_{AB}\rangle = \sum_i \lambda_i |e_i\rangle_A |f_i\rangle_B$ //.

(4) Purification:

Given $\rho_A \rightarrow |\psi\rangle_{AB}$ such that $Tr_B(|\psi_{AB}\rangle\langle\psi|_{AB})$

$$\rho_A = \sum_i p_i |i\rangle\langle i| \rightarrow |\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A |i\rangle_B$$

* Local unitary freedom :-

$(I_A \otimes U_B)|\psi\rangle_{AB}$ is also a valid purification of ρ_A,

where U_B is a unitary on $\mathcal{H}_B$ alone.